CALCULATION OF DE WITT–SCELEY–GILKEY COEFFICIENTS FOR MINIMAL FOURTH–ORDER OPERATOR

I.A.Shovkovy

Bogolyubov Institute for Theoretical Physics,

252143 Kiev, Ukraine

Abstract

De Witt–Sceley–Gilkey coefficients are calculated for the most general minimal differential fourth–order operator on Riemannian space of an arbitrary dimension.

As is well known, the diagonal matrix elements of the heat kernel, $\langle x|\exp(-tA)|x\rangle$ where A is an elliptic differential operator, can be expanded into asimptotic series at $t\to\infty$ [1]. These series have the form:

$$\langle x|\exp(-tA)|x\rangle = \sum_{m=0}^{\infty} E_m(x|A)t^{(m-n)/2r}, \tag{1}$$

where $2r$ is the operator order and coefficients $E_m(x|A)$, called De Witt–Seeley–Gilkey (DWSG) coefficients (sometimes they called by names of Schwinger, Hadamard and Minakshisundaram), depend on the coefficient functions of the operator A and their derivatives.

There are several methods for calculating the asymptotic expansion (1). In this paper we use the method developed in [2] which is based on the theory of pseudodifferential operators. The advantage of this method is its explicit covariance with respect to general coordinate transformations.

We consider the fourth–order minimal operator of the form:

$$\Delta = \Box^2 + B^{\mu\nu\lambda}\nabla_\mu\nabla_\nu\nabla_\lambda + V^{\mu\nu}\nabla_\mu\nabla_\nu + N^\mu\nabla_\mu + X, \tag{2}$$

where ∇_μ is a covariant derivative, including both affine and spinor connection (torsion is absent), coefficient functions $B^{\mu\nu\lambda}$, $V^{\mu\nu}$, N^μ and X are matrices. Such operators are used in quantum gravity with quadratic curvature term in lagrangian [3]. In paper [2] practically the same problem was posed. But, in order to simplify calculations, the term with $B^{\mu\nu\lambda}$ was omitted there. Below we shall present those additional terms in DWSG coefficients for operator (2) that contain tensor $B^{\mu\nu\lambda}$.

We suppose that the tensor $B^{\mu\nu\lambda}$ is symmetric in all indices.

The equation for the amplitude $\sigma(x,x',k;\lambda)$ (for the formalism used here see [2]) is:

$$\Big[(\nabla_\mu + i\nabla_\mu l)(\nabla^\mu + i\nabla^\mu l)(\nabla_\nu + i\nabla_\nu l)(\nabla^\nu + i\nabla^\nu l) +$$
$$+B^{\mu\nu\lambda}(\nabla_\mu + i\nabla_\mu l)(\nabla_\nu + i\nabla_\nu l)(\nabla_\lambda + i\nabla_\lambda l) +$$

Following the paper [2], we seek for the solution to the equation (3) in the form of series over an auxilary parameter ϵ : $\sigma = \sum_{m=0}^{\infty} \epsilon^{4+m}\sigma_m$. After changing $l \to l/\epsilon$ and $\lambda \to \lambda/\epsilon^4$ in the equation, we obtain the following recursion expressions:

$$
\begin{aligned}
((\nabla^\mu l \nabla_\mu l)^2 - \lambda)\sigma_m + A_1\sigma_{m-1} + A_2\sigma_{m-2} \quad & + \\
+A_3\sigma_{m-3} + A_4\sigma_{m-4} \quad & = \quad 0, \qquad (4)
\end{aligned}
$$

where $\sigma_m \equiv 0$ for $m < 0$ by definition and

$$
\begin{aligned}
A_1 \quad = \quad & -i\Big(2\Box l \nabla^\mu l \nabla_\mu l + 4\nabla^\mu l \nabla^\nu l \nabla_\mu \nabla_\nu l + \\
& +4\nabla^\mu l \nabla_\mu l \nabla^\nu l \nabla_\nu + B^{\mu\nu\lambda}\nabla_\mu l \nabla_\nu l \nabla_\lambda l\Big),
\end{aligned}
$$

$$
\begin{aligned}
A_2 \quad = \quad & -\Big((\Box l)^2 + 2\nabla^\mu\nabla^\nu l \nabla_\mu \nabla_\nu l + 2\nabla^\mu l(\nabla_\mu \Box l + \Box \nabla_\mu l) + \\
& +3B^{\mu\nu\lambda}\nabla_\mu \nabla_\lambda l \nabla_\nu l + V^{\mu\nu}\nabla_\mu l \nabla_\nu l + 2\nabla^\mu l \nabla_\mu l \Box + \\
& +4\nabla^\mu l \nabla^\nu l \nabla_\mu \nabla_\nu + (4\Box l \nabla^\mu l + 4\nabla_\nu l \nabla^\mu \nabla^\nu l + \\
& + 4\nabla_\nu l \nabla^\nu \nabla^\mu l + 3B^{\mu\nu\lambda}\nabla_\nu l \nabla_\lambda l\big) \nabla_\mu\Big),
\end{aligned}
$$

$$
\begin{aligned}
A_3 \quad = \quad & i\Big(\Box\Box l + B^{\mu\nu\lambda}\nabla_\mu \nabla_\nu \nabla_\lambda l + V^{\mu\nu}\nabla_\mu \nabla_\nu l + N^\mu \nabla_\mu l + \\
& + \left(2\nabla^\mu \Box l + 2\Box \nabla^\mu l + 3B^{\mu\nu\lambda}\nabla_\nu \nabla_\lambda l + 2V^{(\mu\nu)}\nabla_\nu l\right) \nabla_\mu + \\
+ \quad & 2\Box l \Box + \left(4\nabla^\mu \nabla^\nu l + 3B^{\mu\nu\lambda}\nabla_\lambda l\right) \nabla_\mu \nabla_\nu + \\
& +2\nabla^\mu l \left(\nabla_\mu \Box + \Box \nabla_\mu\right)\Big), \\
A_4 \quad = \quad & \Delta.
\end{aligned}
$$

Using technique of the paper [2], the coefficients $[\sigma_m]$ $(m = 0, 1, 2, 3, 4)$ can be easily found. To avoid unnecessary complications in this paper we do not write down them (for details see [4]). This functions lead to the following expressions for DWSG coefficients:

$$
E_2(x|\Delta) \quad = \quad \frac{1}{(4\pi)^{n/2}} \frac{\Gamma((n-2)/4)}{2\Gamma((n-2)/2)} \Big(\frac{1}{6}R + \frac{1}{2n} V_\alpha^{\ \alpha} - \frac{3}{4n}\nabla_\alpha b^\alpha -
$$

where

$$
\begin{aligned}
h_1 &= \frac{1}{4}\nabla_\alpha B^{\alpha\beta\gamma}R_{\beta\gamma} - \frac{1}{8}\nabla_\alpha b^\alpha R - \frac{1}{4}b_\beta\nabla_\alpha W^{\alpha\beta} + \\
&+ \frac{n}{4(n+2)}\nabla_\alpha\nabla_\beta\nabla_\gamma B^{\alpha\beta\gamma} - \frac{3}{4(n+2)}\left(-\frac{1}{2}\{b_\alpha, N^\alpha\} + \right. \\
&+ \left. V^{\alpha\beta}\nabla^\gamma B_{\alpha\beta\gamma} + b_\alpha\nabla_\beta V^{(\alpha\beta)} + \frac{1}{2}V_\beta{}^\beta\nabla_\alpha b^\alpha\right) + \\
&+ \frac{n+4}{12(n+2)}\nabla_\beta(b_\alpha R^{\alpha\beta} + 3b_\alpha W^{\beta\alpha}) - \\
&- \frac{n+4}{8(n+2)}\{\nabla^\alpha\nabla_\beta\nabla_\alpha\}b^\beta + \frac{n+1}{12(n+2)}[b^\alpha, \nabla^\beta W_{\beta\alpha}] + \\
&+ \frac{1}{4(n+2)}\left([V^{(\alpha\beta)}, \nabla_\alpha b_\beta] + [B^{\alpha\beta\gamma}, \nabla_\alpha V_{\beta\gamma}] - \right. \\
&- \left. [\nabla_\alpha V^{(\alpha\beta)}, b_\beta] - [\nabla_\alpha B^{\alpha\beta\gamma}, V_{\beta\gamma}]\right) + \\
&+ \frac{1}{8(n+2)}\left([b^\alpha, \nabla_\alpha V_\beta{}^\beta] - [\nabla_\alpha b^\alpha, V_\beta{}^\beta]\right),
\end{aligned}
$$

$$
\begin{aligned}
h_2 &= -\frac{\{(B_{\circ\circ\circ})^2, V_{\circ\circ}\} + B_{\circ\circ\circ}V_{\circ\circ}B_{\circ\circ\circ}}{3\cdot 2^5(n+6)(n+2)} - \frac{R(B_{\circ\circ\circ})^2}{96(n+2)} - \\
&- \frac{n+8}{96(n+6)(n+2)}\left(\Box(B_{\circ\circ\circ})^2 - \nabla_\alpha B_{\circ\circ\circ}\nabla^\alpha B_{\circ\circ\circ} + \right. \\
&+ \left. 18B^\alpha_{\circ\circ}\nabla_{(\alpha}\nabla_{\beta)}B^\beta_{\circ\circ}\right) + \frac{3(n+4)}{16(n+6)(n+2)}\left(\nabla_\alpha\nabla_\beta b_\gamma B^{\alpha\beta\gamma} + \right. \\
&+ \left. \nabla_\alpha\nabla_\beta(B^{\alpha\beta\gamma}b_\gamma) + \nabla_{(\alpha}(\nabla_{\beta)}B^\alpha_{\circ\circ}B^\beta_{\circ\circ})\right) + \\
&+ \frac{3(3n+16)}{16(n+6)(n+2)}\left(b_\alpha\nabla_\beta\nabla_\gamma B^{\alpha\beta\gamma} + [\nabla_\alpha b_\gamma, \nabla_\beta B^{\alpha\beta\gamma}]\right) + \\
&+ \frac{3}{16(n+2)}\left(\nabla_\alpha(B^\alpha_{\circ\circ}\nabla_\beta B^\beta_{\circ\circ} - B^\beta_{\circ\circ}\nabla_\beta B^\alpha_{\circ\circ}) - B^\alpha_{\circ\circ}W_{\alpha\beta}B^\beta_{\circ\circ} + \right. \\
&+ \left. 2B^\alpha_{\circ\circ}B^\beta_{\circ\circ}W_{\alpha\beta} + B^\alpha_{\circ\circ}B^\beta_{\circ\circ}R_{\alpha\beta} - 4B^{\alpha\beta\sigma}B_\sigma{}^{\gamma\delta}R_{\alpha\gamma\beta\delta}\right) + \\
&+ \frac{3R_{\alpha\beta}}{16(n+2)}\{B^{\alpha\beta\gamma}, b_\gamma\},
\end{aligned}
$$

$$
\begin{aligned}
h_3 &= \frac{(B_{\circ\circ\circ})^4}{3\cdot 2^9(n+10)(n+6)(n+2)} + \\
&+ 2\nabla_\circ(B_{\circ\circ\circ})^3 + [(B_{\circ\circ\circ})^2, \nabla_\circ B_{\circ\circ\circ}]
\end{aligned}
$$

Since the expression for h_0 does not include terms with $B^{\mu\nu\lambda}$ (and, consequently, coincides with h_0 found in [2]), it is not presented here.

As for the notations, all of them coincide with those in [2], except for the following: $b_\alpha = B_{\alpha\beta}^{\ \ \ \beta}$ and $T_{\circ\circ\ldots\circ} = g^{\{\alpha\beta\ldots\lambda\}}T_{\alpha\beta\ldots\lambda}$.

DWSG coefficients for the operator (2) on a manifold of dimension $n = 4$ were found in [5,6]. In order to check the result obtained here we presented the expressions in the same form as in [5].

Substituting $n = 4$ in $E_2(x|\Delta)$ and $E_4(x|\Delta)$, one can see that, except for three places, our result is the same.

The indicated differences are the following:

a) the expression for h_2, obtained here, contains two additional terms:

$$\tfrac{3}{16(n+2)}\left(B^{\alpha}_{\ \circ\circ}B^{\beta}_{\ \circ\circ}R_{\alpha\beta} - 4B^{\alpha\beta\sigma}B_{\sigma}^{\ \gamma\delta}R_{\alpha\gamma\beta\delta}\right),$$

which are absent in the above mentioned paper. However, these terms are present in the second paper [6], where the result has slightly different form. We can conclude that in [5] there is a mistake.

b) in the expression for h_1, instead of the term

$$(1/4)\nabla_\alpha B^{\alpha\beta\gamma}R_{\beta\gamma},$$

in the mentioned paper we find

$$(1/4)\nabla_\alpha B^{\alpha\beta\gamma}\nabla_\gamma\nabla_\beta R.$$

As in the first case comparison with [6] solves the question again. So we conclud that in [5] there is one more misprint.

c) obtained here expression for h_1 contains one more term which is absent in [5]:

$$-\frac{1}{8(n+2)}\left[\nabla_\alpha b^\alpha, V\right]. \tag{5}$$

themselves to the operators with commuting coefficient functions $B^{\mu\nu\lambda}$ and $V^{\mu\nu}$ and, consequently, all expressions like (5) are identically equal to zero.

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